2021 年秋季学期课程期中考试试卷答题纸

课程名称：高等代数 I　　　　课程代码：MATH120011.05/06

卷　　别：■A 卷　　□B 卷　　□C 卷

姓　　名：________________　　　　学　　号：________________

我已知悉学校与考试相关的纪律以及违反纪律的后果，并将严守纪律，不作弊，不抄袭，独立答题。

学生（签名）：

年　　月　　日

题号	1 (20 分)	2 (20 分)	3 (20 分)	4 (20 分)	5 (20 分)	总分 (100 分)
得分						

注意：

- 请遵守复旦大学考场规定。
- 请用英文或中文答题。
- 书写答案应尽量工整，避免字迹潦草难以辨认。
- 前 6 页为题目及答题纸，请把答案写在前 6 页或其背面。注意保持装订完整。
- 后 4 页为草稿纸，答卷前撕下使用，交卷时应一并上交。
- 计算题只需写下结果，不用写过程。计算结果错误不得分。
- 其余证明题均需提供必要的论证过程及解释。

1.(20 pt) Computations. All matrices and linear equations are over the real number field $\mathbb{R}$. Write down the correct results of each question.

(1) (7 pt) Compute the inverse of $\begin{bmatrix} 3 & -4 & 5 \\ 2 & -3 & 1 \\ 3 & -5 & -1 \end{bmatrix}$;

(2) (6 pt) Compute the rank of $A = \begin{bmatrix} 2 & -1 & 3 & -2 & 4 \\ 4 & -2 & 5 & 1 & 7 \\ 2 & -1 & 1 & 8 & 2 \end{bmatrix}$;

(3) (7 pt) Describe the solutions of

$$A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ 1 \end{bmatrix}.$$

2.(20 pt) Let V, W be finite-dimensional $\mathbb{R}$-linear spaces. Fix a positive integer n. Let $\alpha_1, \ldots, \alpha_n \in V$, and $\beta_1, \ldots, \beta_n \in W$. Prove that the following two statements are **equivalent**:

(1) There is an $\mathbb{R}$-linear map $T : V \to W$ such that $T\alpha_i = \beta_i$ for all $1 \leq i \leq n$.
(2) For any $c_1, \ldots, c_n \in \mathbb{R}$, if $\sum_{i=1}^n c_i\alpha_i = 0$, then $\sum_{i=1}^n c_i\beta_i = 0$.

3.(20 pt) Let V, W be finite-dimensional $\mathbb{R}$-linear spaces and let $T : V \to W$ be a linear map. For a subset $U \subset W$, denote $T^{-1}(U)$ to be the pre-image of U under T, that is, $T^{-1}(U) = \{\alpha \in V \mid T(\alpha) \in U\}$.

(1) (2 pt) If W' is a subspace of W, show that $T^{-1}(W')$ is a subspace of V.

(2) (4 pt) Given 2 subspaces $W_1, W_2 \subset W$. Show that $T^{-1}(W_1 + W_2) = T^{-1}(W_1) + T^{-1}(W_2)$.

(3) (14 pt) Given 2 subspaces $W_1, W_2 \subset W$. Show that the following 2 statements are **equivalent**.

(a) $W_1 \cap W_2 \cap R(T) = \{0\}$ and T is injective.

(b) $\dim(T^{-1}(W_1)) + \dim(T^{-1}(W_2)) = \dim(T^{-1}(W_1 + W_2))$.

注：此题第2小问命题有误。尝试考虑以下两个问题：

1. 找到第2小问的反例；
2. 在第2小问结论成立的前提下，证明第3小问。

4.(20 pt) Given a positive integer n. A matrix $A = [a_{ij}] \in \mathbb{R}^{n\times n}$ is called *a magic matrix* if the sum of entries in each column and each row is a fixed number, namely, $\sum_{k=1}^{n} a_{kj} = \sum_{k=1}^{n} a_{ik}$ for all $1 \le i, j \le n$. A matrix $A \in \mathbb{R}^{n\times n}$ is called *a permutation matrix* if in each column and each row, there is exactly one entry to be 1 and others are 0 (for example, the identity matrix I_n). Denote $\mathcal{M}$ to be the set of all magic matrices in $\mathbb{R}^{n\times n}$, and $\mathcal{P}$ to be the set of all permutation matrices in $\mathbb{R}^{n\times n}$.

(1) (4 pt) Compute $|\mathcal{P}|$.
(2) (8 pt) Compute the dimension of $\mathrm{Span}(\mathcal{M})$.
(3) (8 pt) Show that $\mathrm{Span}(\mathcal{M}) = \mathrm{Span}(\mathcal{P})$.

5.(20 pt) Let V be an $\mathbb{R}$-linear space. Let $T : V \to V$ and $S : V \to V$ be invertible linear maps such that $T^{-1} = T$ and $S^{-1} = S$.

(1) (3 pt) Show that $(T - S)(T + S) = -(T + S)(T - S)$.

(2) (3 pt) Show that $(T + S)^2 + (T - S)^2 = 4I$, where I is the identity map.

(3) (14 pt) Show that

$$R(ST - TS) = R(T - S) \cap R(T + S).$$

Here $R(*)$ is the range of the linear map.