2020 年秋季学期课程期末考试试卷答题纸

课程名称： 高等代数 I 课程代码： MATH120011.04

卷 别：■A 卷 □B 卷 □C 卷

姓 名：________________ 学 号：________________

我已知悉学校与考试相关的纪律以及违反纪律的后果，并将严守纪律，不作弊，不抄袭，独立答题。

学生（签名）：

年 月 日

题号	1 (10 分)	2 (10 分)	3 (15 分)	4 (15 分)	5 (15 分)	6 (17 分)	7 (18 分)	总分 (100 分)
得分								

注意：

- 请遵守复旦大学考场规定。
- 请用英文或中文答题。
- 书写答案应尽量工整，避免字迹潦草难以辨认。
- 前 8 页为题目及答题纸，请把答案写在前 8 页或其背面。
- 后 4 页为草稿纸，可以撕下，交卷时应一并上交。
- 计算题只需写下结果，不用写过程。
- 证明题需提供必要的论证过程。

1. (10 points) Let $A = \begin{bmatrix} -1 & -2 & 0 \\ -1 & 0 & 2 \\ -1 & -1 & 1 \end{bmatrix}$; let $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$; let $\vec{b}_1 = \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix}$ and $\vec{b}_2 = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$;

(1) find all solutions of the system $A\mathbf{x} = \vec{b}_1$;

(2) find all solutions of the system $A\mathbf{x} = \vec{b}_2$.

2. (10 points) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation, given by the 3×3 matrix A:

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix}.$$

Let $\mathfrak{B} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$,

(1) show that $\mathfrak{B}$ is a basis of $\mathbb{R}^3$;

(2) find $[T]_{\mathfrak{B}}$, which is a 3×3 matrix.

3. (15 points)

(1) Let $A = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$, find $\det A$;

(2) let $B = \begin{bmatrix} 0 & 0 & c_0 \\ 1 & 0 & c_1 \\ 1 & 1 & c_2 \end{bmatrix}$, find its characteristic polynomial $\det(xI - B)$;

(3) Find an $n \times n$ matrix C so that its characteristic polynomial

$$\det(xI - C) = x^n + c_{n-1}x^{n-1} + \cdots + c_1 x + c_0,$$

Prove it.

4. (15 points) Let $T : V \to W$ be linear transformation between two finite dimensional $\mathbb{R}$ vector spaces. Show that

$$\dim \mathrm{Null}(T) + \dim \mathrm{Range}(T) = \dim V.$$

5. (15 points) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation given by $A = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$,

(1) we know $c_1 = 1$ is an eigenvalue of A; find the other eigenvalue c_2; further for each eigenvalue c_1 and c_2, find one of its associated eigenvector;

(2) write the minimal polynomial of A as $p(x) = (x - c_1)(x - c_2)$; let $p_i(x) = (x - c_i)$. Show that each $\mathrm{Null}(p_i(x)|_{x=A})$ is the span of one of the eigenvectors found;

(3) Following Theorem 12 of Section 6, we know that there are projections E_1 and $E_2 : \mathbb{R}^2 \to \mathbb{R}^2$ so that

$$T = c_1 E_1 + c_2 E_2.$$

Find the matrix forms A_1 and A_2, of E_1 and E_2, respectively; verify the above identity.

Let A be an $n \times n$ matrix. We know that the row space (resp. column space) of A is the span of the row vectors (resp. column vectors) of A; the row rank of A is the dimension of its row space, same to the column rank. Answer the following question; when asked why, give a brief reason.

6. (17 points) Let A' be derived from A via a sequence of row operations. Answer the following questions:

(1) Are the row spaces of A and A' the same? Do they have identical dimensions? Why?

(2) Are the column spaces of A and A' the same? Do they have identical dimensions? Why?

(3) In case A' is in Echelon form, how to read the row rank of A'? and how to read off the column rank A'?

7. (18 points) Let $T:\mathbb{R}^6\to\mathbb{R}^6$ be a linear transformation, having characteristic polynomial

$$f(x)=(x-1)^3(x+1)^3.$$

Let $W_\pm=\mathrm{Null}\big((T\pm I)^3\big)$. Show that

(1) W_- and W_+ are T invariant, namely $T(W_\pm)\subset W_\pm$;

(2) $\mathbb{R}^6=W_-\oplus W_+$.