2020 年秋季学期课程期中考试试卷答题纸

课程名称：　高等代数 I　　　　课程代码：　MATH120011.04

卷　　别：■A 卷　　□B 卷　　□C 卷

姓　　名：＿＿＿＿＿＿＿＿　　学　　号：＿＿＿＿＿＿＿＿

我已知悉学校与考试相关的纪律以及违反纪律的后果，并将严守纪律，不作弊，不抄袭，独立答题。

学生（签名）：

年　　月　　日

题号	1 (20 分)	2 (20 分)	3 (20 分)	4 (20 分)	5 (20 分)	总分 (100 分)
得分						

注意：

- 请遵守复旦大学考场规定。
- 请用英文或中文答题。
- 书写答案应尽量工整，避免字迹潦草难以辨认。
- 前六页为题目及答题纸，请把答案写在前六页或其背面。
- 后四页为草稿纸，可以撕下，交卷时应一并上交。
- 第一题为计算题，只需写下结果，不用写过程。
- 其余证明题均需提供必要的论证过程。

1.(20 pt) Computations. Write down the correct results of each question.

(1) (5 pt) Compute the inverse of $\begin{bmatrix} 3 & -1 & 2 \\ 1 & 4 & -3 \\ 2 & 2 & 1 \end{bmatrix}$;

(2) (5 pt) Compute the rank of $\begin{bmatrix} 1 & 7 & 7 & 9 \\ 7 & 5 & 1 & -1 \\ 4 & 2 & -1 & -3 \\ -1 & 1 & 3 & 5 \end{bmatrix}$;

(3) (5 pt) Describe the solutions of

$$\begin{aligned} 5x_1 + 3x_2 + 5x_3 + 12x_4 &= 10; \\ 2x_1 + 2x_2 + 3x_3 + 5x_4 &= 4; \\ x_1 + 7x_2 + 9x_3 + 4x_4 &= 2. \end{aligned}$$

(4) (5 pt) Let $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 4 & 3 & 2 & 1 \\ 1 & 4 & 1 & 1 \\ 5 & 1 & 1 & 1 \\ 1 & 1 & 3 & 1 \end{bmatrix}$, find an invertible matrix P such that PA is a row-reduced echelon matrix.

2.(20 pt) Let V be an $\mathbb{R}$-linear space. Consider two finite subsets S, T of V. Suppose that S is linearly independent, and T spans V. Show that for any integer $0 \le k \le \min\{|S|, |T|\}$, we can find subsets $S_k \subseteq S$, $T_k \subseteq T$, such that $|S_k| = |T_k| = k$ and $(T \setminus T_k) \cup S_k$ spans V.

3.(20 pt) Let V be an n-dimensional $\mathbb{R}$-linear space. Consider an $\mathbb{R}$-linear map $T : V \to V$ satisfying $T^2 = -I$.

(1) (5 pt) Show that for any non-zero $\alpha \in V$, $\{\alpha, T\alpha\}$ is linearly independent.

(2) (10 pt) Show that we can find an ordered basis of V of the form

$$\{\alpha_1, T\alpha_1, \alpha_2, T\alpha_2, \ldots, \alpha_k, T\alpha_k\}.$$

(3) (3 pt) Write down the matrix of T relative to the ordered basis in (2).

(4) (2 pt) Show that any two matrices $A, B \in \mathbb{R}^{n\times n}$ satisfying $A^2 = B^2 = -I_n$ are similar to each other.

4.(20 pt) Let m, n, k, l be positive integers. Consider two matrices $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{k\times l}$. Consider the $\mathbb{R}$-linear map $T : \mathbb{R}^{n\times k} \to \mathbb{R}^{m\times l}$ defined by $T(X) = AXB$.

(1) (8 pt) Show that T is invertible if and only if $m = n, k = l$, and A, B are invertible.

(2) (12 pt) Compute $r(T)$ in terms of $r(A)$ and $r(B)$.

5.(20 pt) Let V, W, Z be finite-dimensional $\mathbb{R}$-linear spaces. Consider two linear maps $T : V \to W$ and $U : W \to Z$. Prove that

(1) (10 pt) $r(UT) = r(U)$ if and only if there exists a linear map $S : W \to V$ such that $UTS = U$.

(2) (10 pt) $r(UT) = r(T)$ if and only if there exists a linear map $S' : Z \to W$ such that $S'UT = T$.