

第一部份

2022 春季季考

一元微分与 Riemann 积分

1) 可微性. 若 $\exists E \in \mathbb{R}$, 使得

$$f(x_0 + \Delta x) = f(x_0) + E \cdot \Delta x + o(\Delta x), \text{ as } \Delta x \rightarrow 0,$$

$$E = f'(x_0) \in \mathbb{R}$$

(2) $f(x) \in \mathcal{R}[a, b]$. 若

$$\exists \lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{1}{n} f(x_k) \Delta x_k = \int_a^b f(x) dx \in \mathbb{R},$$

$$|D| = \max_{1 \leq k \leq n} \Delta x_k, \forall x_k \in [x_{k-1}, x_k], k=1, \dots, n$$

(3) $f(x)$ 在 $I \subset \mathbb{R}$ 上一致连续 ($\Rightarrow$).

$$\forall \{\tilde{x}_n\}, \{\hat{x}_n\} \subset I, \text{ 若 } \lim_{n \rightarrow \infty} |\tilde{x}_n - \hat{x}_n| = 0, \text{ 则必有 } \lim_{n \rightarrow \infty} |f(\tilde{x}_n) - f(\hat{x}_n)| = 0.$$

证: (i) 证 $f(x)$ 在 I 上一致连续. 若 $\forall \varepsilon > 0, \exists \delta \varepsilon > 0$, 使得

$$|f(\tilde{x}) - f(\hat{x})| < \varepsilon, \forall \tilde{x}, \hat{x} \in I, |\tilde{x} - \hat{x}| < \delta \varepsilon.$$

现取 $|\tilde{x}_n - \hat{x}_n| \rightarrow 0, \text{ as } n \rightarrow +\infty$, 使得 $|\tilde{x}_n - \hat{x}_n| < \delta \varepsilon, \forall n > N/\delta \varepsilon$,

$$\text{故有 } |f(\tilde{x}_n) - f(\hat{x}_n)| < \varepsilon, \forall n > N/\delta \varepsilon, \text{ 从而 } |f(\tilde{x}_n) - f(\hat{x}_n)| \rightarrow 0, \text{ as } n \rightarrow +\infty.$$

(ii) 假设 $f(x)$ 在 I 上不一致连续. 若 $\exists \varepsilon_0 > 0$, 对 $\forall \delta > 0$,
 $\exists \tilde{x}_n, \hat{x}_n \in I$, 使 $|\tilde{x}_n - \hat{x}_n| < \delta$ 而 $|f(\tilde{x}_n) - f(\hat{x}_n)| > \varepsilon_0$. 取 $\{\tilde{x}_n\}, \{\hat{x}_n\} \subset I$,

$$\text{使 } |\tilde{x}_n - \hat{x}_n| < \delta_n = \frac{1}{n} \rightarrow 0, \text{ as } n \rightarrow +\infty, \text{ 而 } |f(\tilde{x}_n) - f(\hat{x}_n)| > \varepsilon_0.$$

(4) - 设 γ 为 $[a, b]$ 上的弧长 $\gamma \mapsto \begin{bmatrix} x(\gamma) \\ y(\gamma) \end{bmatrix}$, (2)

$\frac{d}{ds} \gamma \mapsto \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix}$, 则 $y(x) = y(x(s))$.

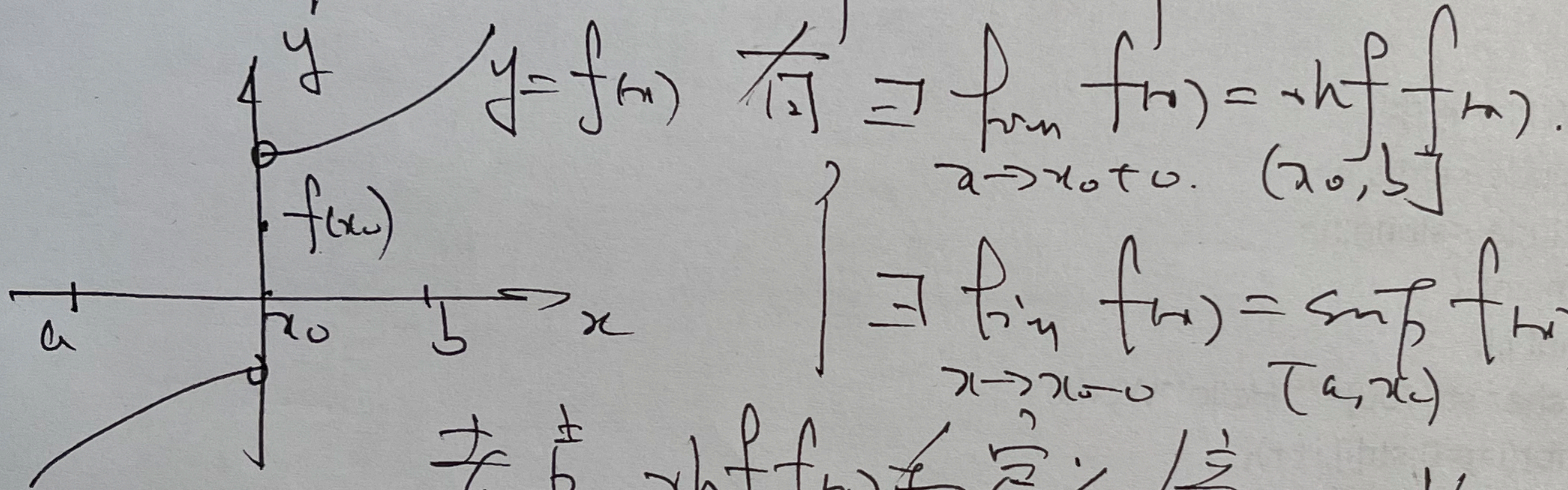
$$\frac{dy}{dx}(x) = \frac{\dot{y}}{\dot{x}}(x(s)), \quad \frac{d^2 y}{dx^2}(x) = \frac{\ddot{y}\dot{x} - \dot{y}\ddot{x}}{\dot{x}^3}(x(s)) \cdot \frac{ds}{dx}(x).$$

$$= \frac{\ddot{y}\dot{x} - \dot{y}\ddot{x}}{\dot{x}^3}(x(s))$$

$$\text{于是 } K = \frac{|\ddot{y}\dot{x} - \dot{y}\ddot{x}|}{|\dot{x}^3|}(x) \cdot \frac{1}{\left(1 + \frac{\dot{y}^2}{\dot{x}^2}(x)\right)^{\frac{3}{2}}} = \frac{|\ddot{y}\dot{x} - \dot{y}\ddot{x}|}{|\dot{x}^3 - \dot{y}^2|^{\frac{3}{2}}}(x).$$

(5). 设 $f(x)$ 在 $(a, b]$ 上连续且有界.

求证: $f(x)$ 在 $(a, b]$ 上可积.



$\frac{d}{ds} \gamma \mapsto \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix}$ 且 $\frac{d^2 y}{dx^2}(x) = \frac{\ddot{y}\dot{x} - \dot{y}\ddot{x}}{\dot{x}^3}(x(s)) \cdot \frac{ds}{dx}(x)$, $\forall \varepsilon > 0, \exists x_\varepsilon \in (x_0, b]$.

$$\inf_{x \in (x_0, b]} f(x) \leq f(x_\varepsilon) < \sup_{x \in (x_0, b]} f(x) + \varepsilon, \text{ 且 } \sqrt{\gamma}.$$

$$\inf_{x \in (x_0, b]} f(x) \leq f(x) \leq f(x_\varepsilon) < \sup_{x \in (x_0, b]} f(x) + \varepsilon, \forall x \in (x_0, x_\varepsilon).$$

(6). 设 $f(x)$ 在 (a, b) 内可微且 $f'(x) \neq 0$.

$f(x) \neq 0$ 在 $(0, 1)$ 内可微且 $f'(x) \neq 0$. 证: $\int_0^1 f(x) dx = 0$.

$$\int_0^1 f(x) dx = 0$$

$f'(x) = \frac{1}{2\sqrt{x}}$ 在 $(0, 1)$ 内可微.

判断: 否.

(7) 设函数 f_n 在有限闭区间 (a, b) 上可微, 且 $\lim_{n \rightarrow \infty} f_n(x) = f(x)$.
 判断: $f(x) = \frac{1}{x}$ 在 $(0, +\infty)$ 上是否有 $f'(x) = -\frac{1}{x^2}$. (3)

$$f_n(x) = -\frac{1}{x^2} + \ln \frac{1}{x} \cdot \frac{1}{x^2}, \quad x_n = \frac{1}{2n + \frac{1}{2}}, \quad \text{且 } f'(x_n) = 0.$$

故: 反例不成立. $f(x) = \frac{1}{x^3}$, $f'(x) = -\frac{1}{x^4} \rightarrow +\infty$ as $x \rightarrow 0$, 且 $f(0+) = 0$

(8). $f(x+T) = f(x)$, 且 $\int_0^1 f(x) dx = 0$, 且 $f(x) \neq 0$.
 $\forall x \in \mathbb{R}$

$$\left| \int_2^x f(t) dt \right| \leq C \in \mathbb{R}^+, \quad \forall x > 2.$$

证: $\frac{1}{x} \downarrow 0$ as $x \rightarrow +\infty$, $\forall \mu \in \mathbb{R}^+$.

Dirichlet-Dirichlet 判别法, 有 $\int_2^{+\infty} \frac{f(x)}{x} dx \in \mathbb{R}$

第一问

(1) $x^2 + \alpha(1 - \cos x) + \beta \ln \frac{1}{x} = x^2 + \alpha \left(\frac{x^2}{2} + o(x^3) \right) + \beta \ln \left(1 - \frac{x^2}{2} + o(x^3) \right).$

$$= \left(1 + \frac{\alpha}{2} \right) x^2 + o(x^3) - \frac{\beta}{2} x^2 + o(x^3).$$

$$= x^2 + \alpha \left(1 + \frac{x^2}{2} - \frac{x^4}{4!} + \frac{x^6}{6!} + o(x^7) \right).$$

$$+ \beta \ln \left(1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + o(x^7) \right),$$

故: $\ln \left(1 - \frac{x^2}{2} + \frac{x^4}{4!} + o(x^5) \right) = -\frac{x^2}{2} + \frac{x^4}{4!} + o(x^5) - \frac{1}{2} \frac{x^4}{(3!)^2} + o(x^5)$

$$= -\frac{1}{2} x^2 + \left(\frac{1}{4!} - \frac{1}{2 \cdot (3!)^2} \right) x^4 + o(x^5).$$

故可令 $x^2: 1 + \frac{1}{2} \alpha - \frac{1}{2} \beta = 0.$

使得 $f(x) = o(x^5)$,

$$x^4: -\frac{\alpha}{4!} + \beta \cdot \left(\frac{1}{4!} - \frac{1}{2 \cdot (3!)^2} \right) = 0.$$

as $x \rightarrow 0$.

解: $\alpha = -\frac{4}{7}$
 $\beta = \frac{30}{7}$

$$(2). f(x) = f(x + \sqrt{x^2 + 1}).$$

(4)

$$f(x) = \frac{1}{x + \sqrt{x^2 + 1}} \cdot \left(1 + \frac{x}{\sqrt{x^2 + 1}}\right) = \frac{1}{\sqrt{x^2 + 1}} = (1 + x^2)^{-\frac{1}{2}}$$

$$= 1 + \sum_{b=1}^{\infty} \binom{-\frac{1}{2}}{b} \cdot x^{2b} + o(x^{2n+1}), \text{ as } x \rightarrow 0.$$

by Taylor.

$$f(x) = x + \sum_{b=1}^{\infty} \binom{-\frac{1}{2}}{b} \frac{x^{2b+1}}{2b+1} + o(x^{2n+2}), \text{ as } x \rightarrow 0,$$

$$\binom{-\frac{1}{2}}{b} = \frac{(-\frac{1}{2})(-\frac{1}{2}-1)\cdots(-\frac{1}{2}-(b-1))}{b!} = (-1)^b \frac{\frac{1}{2} \cdot (\frac{1}{2}+1) \cdots (\frac{1}{2}+(b-1))}{b!}$$

$$= (-1)^b \frac{1 \cdot (1+2) \cdots (1+2(b-1))}{(2b)!!} = (-1)^b \frac{(2b-1)!!}{(2b)!!},$$

$$\text{by Taylor } f(x) = x + \sum_{b=1}^{\infty} (-1)^b \frac{(2b-1)!!}{(2b)!!} \cdot \frac{x^{2b+1}}{2b+1} + o(x^{2n+2}), \text{ as } x \rightarrow 0.$$

$$\frac{1}{(2)}. \lim_{n \rightarrow \infty} \frac{\sum_{b=1}^n \sqrt{(na+b)(na+b+1)}}{n^2} \quad (a > 0).$$

$$\text{S.P. } S_n = \frac{a}{n} \sum_{b=1}^n \sqrt{\left(1 + \frac{b}{na}\right) \left(1 + \frac{b+1}{na}\right)} = \frac{a}{n} \sum_{b=1}^n \sqrt{1 + \frac{2b}{a} + \frac{1}{a^2} \frac{b^2}{n^2} + o(1)}$$

$$= \frac{a}{n} \sum_{b=1}^n \left(1 + \frac{1}{a} \frac{b}{n}\right) \cdot (1 + o(1)) \rightarrow a \cdot \int_0^1 \left(1 + \frac{1}{a} x\right) dx.$$

$$= a \cdot \left(1 + \frac{1}{2a}\right) = a + \frac{1}{2}.$$

$$\frac{2}{(2)}. f(x) = \begin{cases} x^2 \left| \sin \frac{\pi}{x} \right|, & x \neq 0 \\ 0, & x = 0. \end{cases}$$

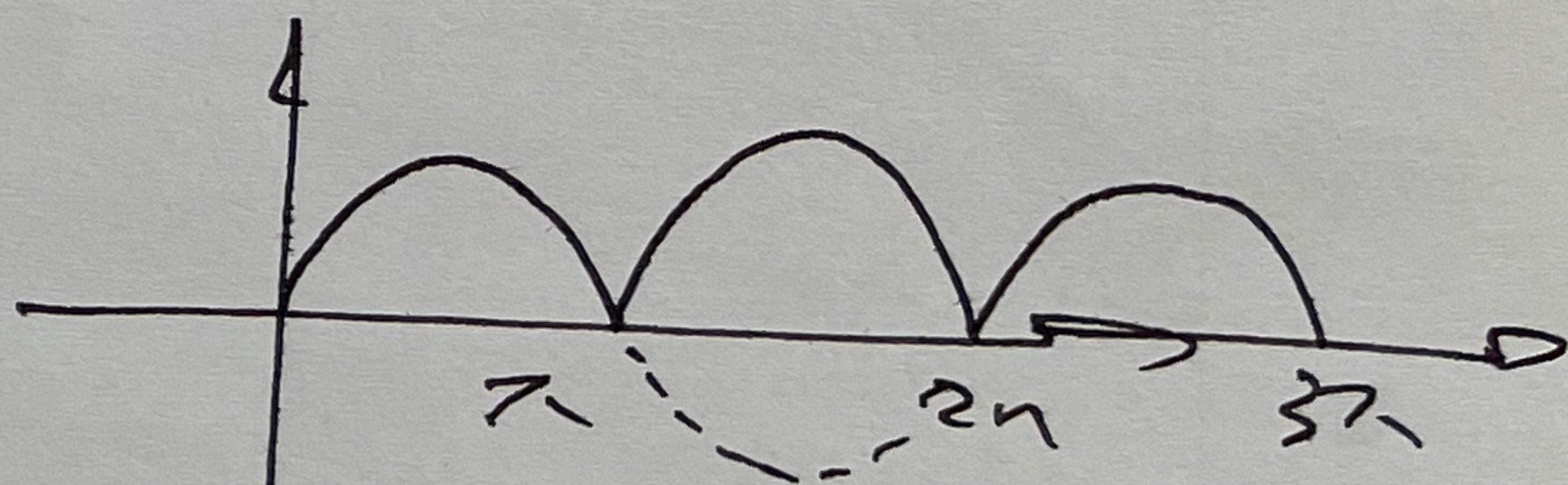
$$\text{S.P. } \frac{f(x) - f(0)}{x} = x \left| \sin \frac{\pi}{x} \right| \rightarrow 0, \text{ by } \exists f'(0) = 0.$$

$$x_n = \frac{1}{n}, \text{ by } f(x_n) = \frac{1}{n^2} \left| \sin \pi n \right| = 0.$$

$$f(x_n + \delta) = \left(\frac{1}{n} + \delta\right)^2 \left| \sin \frac{\pi}{\frac{1}{n} + \delta} \right|$$

$$= \left(\frac{1}{n} + \delta\right)^2 \left| \sin \frac{n\pi}{1 + n\delta} \right| = \left(\frac{1}{n} + \delta\right)^2 \left| \sin \pi n (1 - n\delta + o(\delta)) \right|$$

$$= (\delta^2 + o(\delta)) = \left(\frac{1}{n^2} + o(1)\right) \cdot (-n^2 \delta + o(\delta)), \text{ by } f'_{\pm}\left(\frac{1}{n}\right) = \mp \pi.$$



$$3/17. \int \frac{\ln(x+\sqrt{1+x^2})}{(1+x^2)^{\frac{3}{2}}} dx$$

(5)

$$\text{解: } \int \frac{dx}{(1+x^2)^{\frac{3}{2}}} \stackrel{x=\tan \theta}{=} \int \frac{\sec^2 \theta d\theta}{\sec^3 \theta} = \int \cos \theta d\theta = \sin \theta$$

$$\frac{\sqrt{1+x^2}}{1} \quad x = \frac{x}{\sqrt{1+x^2}}$$

$$I = \int \ln(x+\sqrt{1+x^2}) d \frac{x}{\sqrt{1+x^2}} = \frac{x}{\sqrt{1+x^2}} \ln(x+\sqrt{1+x^2})$$

$$= \int \frac{x}{\sqrt{1+x^2}} \left(1 + \frac{x}{\sqrt{1+x^2}}\right) dx = \frac{x}{\sqrt{1+x^2}} \ln(x+\sqrt{1+x^2})$$

$$- \int \frac{x}{\sqrt{1+x^2}} \frac{1}{x+\sqrt{1+x^2}} \left(1 + \frac{x}{\sqrt{1+x^2}}\right) dx.$$

$$= \frac{x}{\sqrt{1+x^2}} \ln(x+\sqrt{1+x^2}) - \int \frac{x}{1+x^2} dx = \frac{x}{\sqrt{1+x^2}} \ln(x+\sqrt{1+x^2}) - \frac{1}{2} \ln(1+x^2)$$

$$(2). \int_0^{+\infty} \frac{x \ln x}{(1+x^2)^2} dx \stackrel{x=\tan \theta}{=} \int_0^{\frac{\pi}{2}} \tan \theta \ln \tan \theta \cdot \sec^2 \theta d\theta$$

$$\text{解: } \int \frac{x}{(1+x^2)^2} dx = \frac{1}{2} \int \frac{d(1+x^2)}{(1+x^2)^2} = -\frac{1}{2} \cdot \frac{1}{1+x^2}$$

$$I = \int_0^{+\infty} \ln x d\left(-\frac{1}{2} \cdot \frac{1}{1+x^2}\right) = -\frac{1}{2} \left[\ln x \cdot \frac{1}{1+x^2} \right]_0^{+\infty} + \frac{1}{2} \int_0^{+\infty} \frac{1}{1+x^2} \cdot \frac{1}{x} dx$$

由于 \$\frac{1}{x}\$ 在 \$x=0\$ 处不收敛，故不能直接计算，考虑换元。

$$\text{令 } y = \frac{1}{x}, \quad \int_0^{+\infty} \frac{x \ln x}{(1+x^2)^2} dx = \int_{+\infty}^1 \frac{\frac{1}{y} \ln \frac{1}{y}}{\left(1+\frac{1}{y^2}\right)^2} \left(-\frac{1}{y^2}\right) dy$$

$$= - \int_1^{+\infty} \frac{\frac{1}{y} \ln y}{\left(1+y^2\right)^2 \cdot \frac{1}{y^2}} dy = - \int_1^{+\infty} \frac{y \ln y}{(1+y^2)^2} dy$$

故 \$I=0\$.

$$(3). \int_0^{+\infty} \frac{x-1}{(1+x^2)(1+x)} dx$$

(6)

$$\text{Ans. } \frac{1}{(1+x^2)(1+x)} = \frac{A x + B}{1+x^2} + \frac{C}{1+x}, \text{ Eq. } (Ax+B)(1+x) + C(1+x^2)$$

$$= Ax^2 + (A+B)x + B + Cx^2 + C, \text{ Eq. } x^2: A+C=0.$$

$$x^1: A+B=1.$$

$$x^0: B+C=-1.$$

$$\text{Ans. } \begin{cases} A=1 \\ B=0 \\ C=-1 \end{cases}$$

$$\text{Ans. } \int_0^{+\infty} \frac{x-1}{(1+x^2)(1+x)} dx = \int_0^{+\infty} \left(\frac{x}{1+x^2} - \frac{1}{1+x} \right) dx$$

$$= \left[\frac{1}{2} \ln(1+x^2) - \ln(1+x) \right]_0^{+\infty} = \ln \frac{\sqrt{1+x^2}}{1+x} \Big|_0^{+\infty} = 0 - 0 = 0$$

$$\text{Ans. } \int_0^{+\infty} \frac{x-1}{(1+x^2)(1+x)} dx \stackrel{y=\frac{1}{x}}{=} \int_{+\infty}^1 \frac{\frac{1}{y}-1}{(1+\frac{1}{y^2})(1+\frac{1}{y})} \cdot \left(-\frac{1}{y^2}\right) dy$$

$$= \int_1^{+\infty} \frac{1-y}{(1+y^2)(1+y)} dy$$

$$= \int_1^{+\infty} \frac{0}{(1+y^2)(1+y)} dy = 0$$

$$(4) \int_0^{2\pi} x^2 d\theta = \frac{1}{2} \int_0^{2\pi} (1+2\cos\theta+\cos^2\theta) d\theta$$

$$= \frac{1}{2} \int_0^{2\pi} \left(1+2\cos\theta + \frac{1+\cos 2\theta}{2} \right) d\theta = \frac{1}{2} \cdot \frac{3}{2} \cdot 2\pi = \frac{3}{2}\pi$$

$$\begin{aligned}
 4) \text{ Let } I_n &= \int_0^1 (1 + \sinh(\frac{x}{2}))^n dx = \frac{2}{n} \int_0^1 (1 + \sinh(\frac{x}{2}))^n d(\frac{x}{2}) \\
 &= \frac{2}{n} \int_0^{\frac{\pi}{2}} (1 + \sinh y)^n dy \geq \frac{2}{n} \int_0^{\frac{\pi}{2}} (1 + \frac{2}{n} y)^n dy \quad (7) \\
 &= \int_0^{\frac{\pi}{2}} (1 + \frac{2}{n} y)^n d(1 + \frac{2}{n} y) = \frac{(1 + \frac{2}{n} y)^{n+1}}{n+1} \Big|_0^{\frac{\pi}{2}} = \frac{2^{n+1} - 1}{n+1}, \forall n \in \mathbb{N}
 \end{aligned}$$

$$(2) \left(\frac{2^{n+1} - 1}{n+1} \right)^{\frac{1}{n}} = \frac{(2^{n+1} - 1)^{\frac{1}{n}}}{(n+1)^{\frac{1}{n}}} = \frac{2(2 - \frac{1}{2^n})^{\frac{1}{n}}}{(n+1)^{\frac{1}{n}}} \rightarrow 2, \text{ as } n \rightarrow \infty$$

$$\text{So, } \left[\int_0^1 (1 + \sinh(\frac{x}{2}))^n dx \right]^{\frac{1}{n}} \leq 2.$$

to show it is not 2.

$$5. \int_0^{\infty} \frac{\sinh x \cdot f_n(1+x^\beta)}{x^\gamma} dx, \alpha, \beta, \gamma \in \mathbb{R}^+.$$

1st: (1) $x \rightarrow \infty$.

$$f_n(x) = \frac{x^{\alpha+\beta} \cdot (1+x^\beta)}{x^\gamma} = \frac{1}{x^{\gamma-\alpha-\beta}} \cdot (1+x^\beta), \text{ as } x \rightarrow \infty$$

$$\text{2nd: } \begin{cases} \gamma - \alpha - \beta < 1, & (\text{divergent}) \\ \gamma - \alpha - \beta \geq 1, & \text{convergent} \end{cases}$$

$$(2) y = x^\alpha, x = y^{\frac{1}{\alpha}},$$

$$\begin{aligned}
 I &= \int_1^{\infty} \frac{\sinh x \cdot f_n(1+x^\beta)}{x^\gamma} dx = \int_1^{\infty} \frac{\sinh y \cdot f_n(1+y^{\frac{\beta}{\alpha}})}{y^{\frac{\gamma}{\alpha}}} \cdot \frac{1}{\alpha} y^{\frac{1}{\alpha}-1} dy \\
 &= \frac{1}{\alpha} \int_1^{\infty} \frac{\sinh y \cdot f_n(1+y^{\frac{\beta}{\alpha}})}{y^{1+\frac{\gamma-1}{\alpha}}} dy = \frac{1}{\alpha} \int_1^{\infty} g(y) dy
 \end{aligned}$$

(i) to show it is not 1.

$$|g(y)| \leq \frac{f_n(1+y^{\frac{\beta}{\alpha}})}{y^{1+\frac{\gamma-1}{\alpha}}} \leq \frac{C \cdot y^\mu}{y^{1+\frac{\gamma-1}{\alpha}}} = \frac{C}{y^{1+\frac{\gamma-1}{\alpha}-\mu}}, \forall \mu \in \mathbb{R}^+.$$

यदि. $\frac{1}{2} + \frac{\alpha-1}{2} > 1$, तब $\frac{\alpha-1}{2} > 0$, तब $\alpha > 1$. (इसलिए)

(ii) इसलिए (5)

$$\frac{f_n(1+y^{\frac{\beta}{\alpha}})}{y^{1+\frac{\alpha-1}{2}}} \downarrow 0, \text{ as } y \rightarrow +\infty, \frac{1}{2} + \frac{\alpha-1}{2} > 0.$$

तब. $1 + \frac{\alpha-1}{2} > 0$, तब $\alpha + \alpha - 1 > 0$, इसलिए

(iii) इसलिए

$$|g(y)| \geq \frac{f_n(1+y^{\frac{\beta}{\alpha}})}{y^{1+\frac{\alpha-1}{2}}} \text{ तब } = \frac{f_n(1+y^{\frac{\beta}{\alpha}})}{y^{1+\frac{\alpha-1}{2}}} \frac{1-\alpha^2 y}{2},$$

तब $0 < 1 + \frac{\alpha-1}{2} \leq 1$. इसलिए

तब, यदि $1 + \frac{\alpha-1}{2} > 1$, इसलिए, $\frac{\alpha-1}{2} > 0$, तब $\alpha > 1$.

$\wedge 0 < 1 + \frac{\alpha-1}{2} \leq 1$, इसलिए, तब $1 < \alpha \leq 1$

$1 + \frac{\alpha-1}{2} \leq 0$, तब तब $\alpha \leq 1 - \alpha$.

$$\int_1^{+\infty} \frac{x \ln x \cdot f_n(1+x^{\frac{\beta}{\alpha}})}{x^2} dx.$$

$$6. (1) f(x) \in C(-1, +\infty), f(x) \cdot \left[\int_0^x f(t) dt + 1 \right] = \frac{x e^x}{2(1+x)^2}$$

$$\text{इसलिए } F(x) := \int_0^x f(t) dt, \text{ तब } \frac{dF}{dx}(x) = f(x), \text{ तब.}$$

$$\frac{dF}{dx}(x) \cdot [F(x) + 1] = \frac{x e^x}{2(1+x)^2} = \frac{d}{dx} \frac{F(x)^2}{2} - \int \frac{e^x}{(1+x)^2} dx$$

$$\text{इसलिए } F(x)^2 = \int \frac{x e^x}{(1+x)^2} dx = \int \frac{(1+x-1)e^x}{(1+x)^2} dx = \int \frac{e^x}{1+x} dx$$

$$= \int \frac{e^x}{1+x} dx + \int e^x d\left(\frac{1}{1+x}\right) = \frac{e^x}{1+x}$$

$$\text{तब } F(x) = \int_0^x f(t) dt + 1 = \sqrt{\frac{e^x}{1+x}}, \text{ तब}$$

$$f(x) = \frac{1}{2} \cdot \frac{1}{\sqrt{\frac{e^x}{1+x}}} \cdot \frac{e^x(1+x) - e^x}{(1+x)^2} = \frac{1}{2} \cdot \frac{1}{\sqrt{\frac{e^x}{1+x}}} \cdot \frac{e^x \cdot x}{(1+x)^2}$$

$$\frac{b}{(2)} \cdot (x-2) \frac{dy}{dx}(x) + y(x) = 2 \cdot (x-2)^2. \quad (9)$$

$$\frac{dy}{dx}(x) = -\frac{1}{x-2} y(x) + 2(x-2).$$

Step 1. 设 $\bar{y} = \frac{y}{x-2}$, 则 $\frac{d\bar{y}}{dx}(x) = -\frac{1}{x-2} \bar{y}(x).$

则 $\bar{y}(x) = \frac{C}{x-2}$

Step 2. 设 $y(x) = \frac{C(x)}{x-2}$, 则 $\frac{dC}{dx}(x) \cdot \frac{1}{x-2} = 2(x-2).$

则 $\frac{dC}{dx}(x) = 2(x-2)^2$, 则 $C(x) = 2 \int (x-2)^2 dx = \frac{2}{3} (x-2)^3 + C$

故 $y(x) = \frac{1}{x-2} \cdot \left[\frac{2}{3} (x-2)^3 + C \right] = \frac{2}{3} (x-2)^2 + \frac{C}{x-2}.$

第三部分

(1). $\frac{d}{dx} \int_0^x f(x) dx = f(x) = f(0) + f'(0) \cdot x + \frac{f''(0)}{2} x^2 + \frac{f^{(3)}(0)}{6} x^3 + o(x^3),$
as $x \rightarrow 0.$

则 $\int_0^x f(x) dx = f(0)x + \frac{f'(0)}{2} x^2 + \frac{f''(0)}{6} x^3 + \frac{f^{(3)}(0)}{24} x^4 + o(x^4),$ as $x \rightarrow 0.$

$f(\xi(x)) \cdot x = x \cdot \left[f(0) + f'(0) \cdot \xi(x) + \frac{f''(0)}{2} \xi(x)^2 + \frac{f^{(3)}(0)}{6} \xi(x)^3 + \xi(x) \cdot o(1) \right]$

$= f(0) \cdot x + f'(0) \cdot x \cdot \xi(x) + \frac{f''(0)}{2} \cdot x \cdot \xi(x)^2 + \frac{f^{(3)}(0)}{6} x \cdot \xi(x)^3 + x \cdot \xi(x)^3 \cdot o(1).$

$f(0) = f'(0) = 0, f^{(3)}(0) \neq 0, \text{ 则 } \bar{f}.$

$\frac{f^{(3)}(0)}{24} x^4 + o(x^4) = \frac{f^{(3)}(0)}{6} x \cdot \xi(x)^3 + x \cdot \xi(x)^3 \cdot o(1),$ 则

$\frac{1}{24} + o(1) = \frac{1}{6} \frac{\xi(x)^3}{x^3} + \frac{\xi(x)^3}{x^3} \cdot o(1), |\xi(x)| \leq |x|,$

则 $\frac{\xi(x)^3}{x^3} \rightarrow \frac{1}{4},$ as $x \rightarrow 0+0.$

则 $C = \frac{1}{4^{\frac{1}{3}}}, \mu = 3.$

$$(12) / (a) \frac{2}{3} \geq \left(\int_0^1 f(x) dx \right)^2 \geq \int_0^1 f^3(x) dx, \forall x \in (0, 1]. \quad (10)$$

$$\Phi(x) = \left(\int_0^1 f(x) dx \right)^2 - \int_0^1 f^3(x) dx, \quad \Phi(0) = 0.$$

$$\Phi'(x) = 2 \int_0^1 f(x) dx \cdot f(x) - f^3(x) = \left[2 \int_0^1 f(x) dx - f^2(x) \right] \cdot f(x).$$

$$\text{由 } f(x) > 0, \forall x \in (0, 1].$$

$$2f(x) = 2 \int_0^1 f(x) dx - f^2(x), \quad \Phi(0) = 0,$$

$$2f(x) = 2f(x) - 2f(x)f'(x) = 2f(x) \cdot (1 - f'(x)) > 0, \forall x \in (0, 1]$$

$$\text{故 } \Phi'(x) > 0, \forall x \in (0, 1], \text{ 从而 } \Phi(x) > 0, \forall x \in (0, 1].$$

$$\text{即 } \frac{2}{3} \geq \left(\int_0^1 f(x) dx \right)^2 > \int_0^1 f^3(x) dx$$

$$(b) \text{ 设 } f(x) \in C^1[a, b], \text{ 且 } f(a) = 0.$$

$$\int_a^b f^2(x) dx = \int_a^b f(x) \cdot \int_a^x f'(t) dt \cdot dx = \left(\int_a^b f(x) dx \right)^2$$

$$\leq \int_a^b f(x) \cdot \sqrt{\int_a^b (f'(t))^2 dt} \cdot \sqrt{x-a} dx = \sqrt{\int_a^b (f'(t))^2 dt} \cdot \int_a^b f(x) \cdot \sqrt{x-a} dx$$

$$\leq \sqrt{\int_a^b (f'(t))^2 dt} \cdot \sqrt{\int_a^b f^2(x) dx} \cdot \sqrt{\int_a^b (x-a) dx}$$

$$= \sqrt{\int_a^b (f'(t))^2 dt} \cdot \sqrt{\int_a^b f^2(x) dx} \cdot \sqrt{\frac{(b-a)^2}{2}}$$

$$\text{从而 } \int_a^b f^2(x) dx \leq \frac{(b-a)^2}{2} \cdot \int_a^b (f'(x))^2 dx$$

$$3) f_m \in C[0,1], \int_0^1 f_m dx = 0, \int_0^1 x f_m dx = 1. \quad (1)$$

~~$$\frac{1}{2} \leq \frac{1}{2}. \int_0^1 f_m (\alpha + \beta x) dx = \beta = f(\frac{1}{2}) \cdot \int_0^1 (\alpha + \beta x) dx \\ = f(\frac{1}{2}) \cdot (\alpha + \frac{\beta}{2})$$~~

~~$$\text{hence } f(\frac{1}{2}) = \frac{\beta}{\alpha + \frac{\beta}{2}} \Rightarrow \frac{\beta}{\frac{\beta}{2}} = 2, \quad \forall \alpha, \beta \geq 0$$~~

~~$$\frac{1}{5} \leq \frac{1}{5}. \int_0^1 (x-h) f_m dx = 1$$~~

$$|f_1| \leq \int_0^1 |x-h| \cdot |f_m| dx = |f(\frac{1}{2})| \cdot \int_0^1 |x-h| dx, \quad h \in (0,1)$$

$$\text{Q.E.D. } \int_0^1 |x-h| dx = \int_0^h (h-x) dx + \int_h^1 (x-h) dx \\ = h^2 - \frac{h^2}{2} + \frac{1-h^2}{2} - h(1-h) = \frac{(h-1)^2 + h^2}{2}$$

$$\text{hence } |f(\frac{1}{2})| \geq \frac{2}{(h-1)^2 + h^2} \stackrel{h=\frac{1}{2}}{=} 4.$$

由 $|f(\frac{1}{2})| = 0$ 知 f 在 $\frac{1}{2}$ 处不可导。